

Multi-Centered First Order Formalism

Sergio Ferrara^{1,3}, Alessio Marrani¹,
Andrey Shcherbakov³, Armen Yeranyan^{2,3,4}

¹*Physics Department, Theory Unit, CERN,
CH 1211, Geneva 23, Switzerland
sergio.ferrara@cern.ch
alessio.marrani@cern.ch*

²*Museo Storico della Fisica e Centro Studi e Ricerche “ Enrico Fermi”
Via Panisperna 89A, I-00184 Roma, Italy*

³*INFN - Laboratori Nazionali di Frascati,
Via Enrico Fermi 40, I-00044 Frascati, Italy
ashcherb@lnf.infn.it
ayeran@lnf.infn.it*

⁴*Department of Physics, Yerevan State University
Alex Manoogian St. 1, Yerevan, 0025, Armenia*

Abstract

We propose a first order formalism for multi-centered black holes with flat tree-dimensional base-space, within the *stu* model of $N = 2$, $D = 4$ ungauged Maxwell-Einstein supergravity. This provides a unified description of first order flows of this universal sector of all models with a symmetric scalar manifold which can be obtained by dimensional reduction from five dimensions.

We develop a $D = 3$ Cartesian formalism which suitably extends the definition of central and matter charges, as well as of black hole effective potential and first order “fake” superpotential, in order to deal with not necessarily axisymmetric solutions, and thus with multi-centered and/or (under-)rotating extremal black holes.

We derive general first order flow equations for composite non-BPS and almost BPS classes, and we analyze some of their solutions, retrieving various single-centered (static or under-rotating) and multi-centered known systems.

As in the t^3 model, the almost BPS class turns out to split into two general branches, and the well known almost BPS system is shown to be a particular solution of the second branch.

1 Introduction

Multi-centered extremal black hole (BH) solutions in four-dimensional supergravity theories have been widely investigated in recent years [1, 2, 3, 4, 5, 6, 7].

Motivated by the issue of matching the true BPS spectrum and the spectrum of spherically symmetric BHs in supergravity, in [8] a supersymmetric class of multi-centered BHs was introduced, whose BPS first order flow equations were then solved and analyzed in [9], revealing interesting features, such as fixed distance among the centers (in presence of *mutually non-local* electric-magnetic charge vectors).

However, only in the last two years a group-theoretical approach [1, 2], based on nilpotent orbits and timelike reduction to three dimensions, allowed for a systematic construction and investigation of whole new classes of solutions, in which some or all BH centers are non-supersymmetric (non-BPS).

An elegant approach to the flow dynamics of scalar fields in the background of single-centered extremal BH solutions of Maxwell-Einstein theories of (super)gravity, essentially based on the first order reformulation of the scalar equations of motion, was introduced in [10], and then developed in various works [11, 12, 13]. The possibility to switch from second order to first order differential equations of motion - *without doubling their number* - has an applicative relevance. Indeed, due to the interplay between auxiliary fields and scalar charges, the first order formalism automatically discards blowing-up solutions; furthermore, the integration of first order equations is surely more manageable, and explicit forms of attractor flows can be more easily determined.

The extension of such a formalism to non-supersymmetric multi-centered configurations was started in [7] by reducing the relevant action to a sum of squares, however without yielding explicit expressions for the flow equations and their corresponding governing functions. A consistent and explicit determination of the most general first order flow equations for non-BPS multi-centered and/or rotating BHs with flat three-dimensional base-space was achieved in [14], within the simplest model of $N = 2$, $D = 4$ Maxwell-Einstein ungauged supergravity with a cubic prepotential, namely the so-called t^3 model, exhibiting only one vector multiplet, and whose uplift to $D = 5$ is “*pure*” minimal supergravity (see *e.g.* [15]).

In the present investigation, we further develop the approach of [14], and determine the general first order flow equations for all classes of multi-centered BH solutions with flat three-dimensional base-space in the so-called *stu* model [16]; in this model, three vector multiplets are coupled, in a *triality-invariant* way, to the $N = 2$ gravity multiplet, and the resulting completely factorized rank-3 symmetric special Kähler manifold $[SU(1, 1)/U(1)]^3$ can be considered a universal sector of all symmetric scalar manifolds of $D = 4$, $N \geq 2$ -extended supergravity theories which admit a $D = 5$ uplift.

A key result is the reformulation of second order equations of motion in a manifestly $D = 3$ Cartesian formalism, based on a timelike Lagrangian reduction $D = 4 \rightarrow 3$ in a stationary BH background [17], later specialized for flat spatial slices. This formalism allows for a consistent generalization of the BH effective potential [18], and of its expression in terms of supersymmetry central charges and matter charges, as well as in terms of a first order “fake” superpotential [10], in not necessarily axisymmetric contexts, which thus can include multi-centered solutions.

Various under-rotating stationary (BPS and non-BPS) single-centered solutions, as well as the known classes of *BPS* [8], *composite non-BPS* [1] and *almost BPS* [1, 3] multi-centered solutions (possibly constrained in a suitable way) are retrieved as particular solutions of the various first order systems which we propose. It should be stressed that, analogously to [14],

in this paper we focus on the general formulation of first order systems of equations, leaving the determination of new sets of solutions and the investigation of their physical properties to future investigations.

The paper is organized as follows.

In Sec. 2, we start with the timelike reduction of Maxwell-Einstein-scalar action in the background of stationary metrics (then assumed to have flat spatial slices). As mentioned, this yields to a $D = 3$ Cartesian formalism, which is especially suitable to handle multi-centered and under-rotating single-centered solutions.

Our general approach to first order formalism is then exploited in Sec. 3 for the various classes of multi-centered solutions, namely the *BPS* (Sec. 3.1.1), *almost BPS* (Sec. 3.1.2) and *composite non-BPS* (Sec. 3.1.3) classes, also retrieving and discussing known multi- and single-centered solutions. In particular, as found in [14] in the t^3 model, we find that the *almost BPS* class splits into two branches, which can essentially be related to the BPS or non-BPS nature of the corresponding single-centered limits, as discussed in Secs. 5.1 and 5.2.

Finally, Sec. 6 contains a summary and an outlook of results.

Some notation and useful formulæ are given in App. A.

2 Second Order Equations of Motion

Let us consider the $D = 4$ Einstein-Maxwell action

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} R + G_{a\bar{a}} g^{\mu\nu} \partial_\mu z^a \partial_\nu \bar{z}^{\bar{a}} + \frac{1}{4} \mu_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} + \frac{1}{4} \nu_{\Lambda\Sigma} F_{\mu\nu}^\Lambda * F^{\Sigma\mu\nu} \right], \quad (2.1)$$

which, depending on properties of the target space metric $G_{a\bar{b}}(z, \bar{z})$ and of the coupling matrices $\mu_{\Lambda\Sigma}(z, \bar{z})$ and $\nu_{\Lambda\Sigma}(z, \bar{z})$, may be the bosonic sector of some $N \geq 2$ -extended, $D = 4$ Maxwell-Einstein supergravity. In the present investigation, we are interested in stationary solutions to the Einstein-Maxwell equations of motion; we will thus perform a timelike reduction to tree dimensions *à la Breitenlohner-Gibbons-Maison* [17], giving rise to a $D = 3$ Cartesian formalism, which is not necessarily axisymmetric.

We start from a space-time metric satisfying the following *Ansatz* ($i, j = 1, 2, 3$):

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{2U(x)} (dt + \omega_i(x) dx^i)^2 - e^{-2U(x)} \gamma_{ij}(x) dx^i dx^j. \quad (2.2)$$

In this background, the Abelian vector potential A_μ^Λ (defining the two-form field strength¹ $F_{\mu\nu}^\Lambda = 2\partial_{[\mu} A_{\nu]}^\Lambda$) splits into $D = 3$ vector potentials a_i^Λ and Kaluza-Klein scalars b^Λ :

$$A_\mu^\Lambda dx^\mu = b^\Lambda (dt + \omega_i dx^i) + a_i^\Lambda dx^i.$$

By using the $D = 3$ vectors a_i^Λ and ω_i , one can construct the corresponding field strengths

$$f_{ij}^\Lambda = 2\partial_{[i} a_{j]}^\Lambda, \quad W_{ij} = 2\partial_{[i} \omega_{j]},$$

respectively expressing the magnetic field and the rotation in tree dimensions.

¹Throughout the paper, the (anti)symmetrization is defined with a $1/n!$ normalization if applied to n indices.

On stationary background (2.2), the action (2.1) is equivalent to the following $D = 3$ one:

$$S = \int d^3x \sqrt{\gamma} \left[-\frac{1}{2} R^{(3)} + G_{a\bar{a}} \gamma^{ij} \partial_i z^a \partial_j \bar{z}^{\bar{a}} + \gamma^{ij} \partial_i U \partial_j U - \frac{1}{8} e^{4U} W_{ij} W^{ij} \right. \\ \left. + \frac{1}{2} e^{-2U} \mu_{\Lambda\Sigma} \gamma^{ij} \partial_i b^\Lambda \partial_j b^\Sigma - \frac{1}{4} e^{2U} \mu_{\Lambda\Sigma} (f_{ij}^\Lambda + b^\Lambda W_{ij}) (f^{\Sigma ij} + b^\Sigma W^{ij}) \right. \\ \left. - \nu_{\Lambda\Sigma} \partial_i b^\Lambda (*f^{\Sigma i} + b^\Sigma *W^i) \right], \quad (2.3)$$

where the $D = 3$ Hodge operator “ $*$ ” is defined with respect to the 3-dimensional metric γ_{ij} :

$$*A^i \equiv \frac{1}{2\sqrt{\gamma}} \varepsilon^{ijk} A_{jk}.$$

The $D = 3$ two-forms f_{ij}^Λ and W_{ij} can be dualized into scalars b_Λ and ψ by adding the following terms to the action (2.3)

$$\int d^3x \left[\frac{1}{2} b_\Lambda \varepsilon^{ijk} \partial_i f_{jk}^\Lambda + \frac{1}{2} \psi \varepsilon^{ijk} \partial_i W_{jk} \right],$$

thus allowing for the two-forms to be expressed in terms of their duals:

$$f^{\Lambda ij} + b^\Lambda W^{ij} = -\frac{1}{\sqrt{\gamma}} e^{-2U} \mu^{\Lambda\Sigma} \varepsilon^{ijk} \left(\partial_k b_\Sigma + \nu_{\Sigma\Sigma'} \partial_k b^{\Sigma'} \right); \quad (a) \\ W^{ij} = -\frac{2}{\sqrt{\gamma}} e^{-4U} \varepsilon^{ijk} \left(\partial_k \psi - b^\Lambda \partial_k b_\Lambda \right), \quad (b) \quad (2.4)$$

as well as the action (2.3) to be recast as:

$$S = \int d^3x \sqrt{\gamma} \left[-\frac{1}{2} R^{(3)} + G_{a\bar{a}} \gamma^{ij} \partial_i z^a \partial_j \bar{z}^{\bar{a}} + \gamma^{ij} \partial_i U \partial_j U + \frac{1}{2} e^{-2U} \mu_{\Lambda\Sigma} \gamma^{ij} \partial_i b^\Lambda \partial_j b^\Sigma \right. \\ \left. + \frac{1}{2} e^{-2U} \mu^{\Lambda\Sigma} \gamma^{ij} \left(\partial_i b_\Lambda + \nu_{\Lambda\Lambda'} \partial_i b^{\Lambda'} \right) \left(\partial_j b_\Sigma + \nu_{\Sigma\Sigma'} \partial_j b^{\Sigma'} \right) \right. \\ \left. + e^{-4U} \gamma^{ij} \left(\partial_i \psi - b^\Lambda \partial_i b_\Lambda \right) \left(\partial_j \psi - b^\Sigma \partial_j b_\Sigma \right) \right]. \quad (2.5)$$

By grouping the Kaluza-Klein scalars b^Λ and the scalars b_Λ (dual to f_{ij}^Λ) into a symplectic vector b^α (with α running over contravariant and covariant symplectic index Λ) and introducing a scalar $\tilde{\psi}$

$$b^\alpha \equiv \begin{pmatrix} b^\Lambda \\ b_\Lambda \end{pmatrix}, \quad \tilde{\psi} \equiv 2\psi - b^\Lambda b_\Lambda, \quad (2.6)$$

the action (2.5) can be rewritten in more compact form:

$$S = \int d^3x \sqrt{\gamma} \left[-\frac{1}{2} R^{(3)} + G_{a\bar{a}} \gamma^{ij} \partial_i z^a \partial_j \bar{z}^{\bar{a}} + \gamma^{ij} \partial_i U \partial_j U + \frac{1}{2} e^{-2U} \gamma^{ij} \partial_i b^\alpha M_{\alpha\beta} \partial_j b^\beta \right. \\ \left. + \frac{1}{4} e^{-4U} \gamma^{ij} \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right) \left(\partial_j \tilde{\psi} + \langle b, \partial_j b \rangle \right) \right]. \quad (2.7)$$

For the definitions of the symplectic symmetric matrix $M_{\alpha\beta}$ and the skew-symmetric symplectic product $\langle \cdot, \cdot \rangle$, see App. A. Then, by introducing a symplectic vector $f_{ij}^\alpha = (f_{ij}^\Lambda, f_{\Lambda ij})^\top$

constructed from the field strengths, Eqs. (2.4) enjoy a manifest symplectic covariance²:

$$\begin{aligned} f_{ij}^\alpha &= \sqrt{\gamma} \epsilon_{ijk} \gamma^{kk'} e^{-2U} M^{\alpha\beta} \Omega_{\beta\gamma} \partial_{k'} b^\gamma - b^\alpha W_{ij}; & (a) \\ W_{ij} &= -\sqrt{\gamma} e^{-4U} \epsilon_{ijk} \gamma^{kl} \left(\partial_l \tilde{\psi} + \langle b, \partial_l b \rangle \right). & (b) \end{aligned} \quad (2.8)$$

The second order equations of motion pertaining to the action (2.7) consist of the Einstein equations

$$\begin{aligned} \frac{1}{2} R_{ij}^{(3)} &= G_{a\bar{a}} \partial_{(i} z^a \partial_{j)} \bar{z}^{\bar{a}} + \partial_i U \partial_j U + \frac{1}{2} e^{-2U} \partial_i b^\alpha M_{\alpha\beta} \partial_j b^\beta \\ &\quad + \frac{1}{4} e^{-4U} \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right) \left(\partial_j \tilde{\psi} + \langle b, \partial_j b \rangle \right), \end{aligned} \quad (2.9)$$

and of the following ones:

$$\begin{aligned} \partial_i \left[\sqrt{\gamma} \gamma^{ij} e^{-4U} \left(\partial_j \tilde{\psi} + \langle b, \partial_j b \rangle \right) \right] &= 0, \\ \partial_i \left[\sqrt{\gamma} \gamma^{ij} e^{-2U} M_{\alpha\beta} \partial_j b^\beta \right] &= \partial_i \left[\sqrt{\gamma} \gamma^{ij} e^{-4U} \Omega_{\alpha\beta} b^\beta \left(\partial_j \tilde{\psi} + \langle b, \partial_j b \rangle \right) \right], \\ \frac{1}{\sqrt{\gamma}} \partial_i \left[\sqrt{\gamma} \gamma^{ij} \partial_j U \right] &= -\frac{1}{2} e^{-2U} \gamma^{ij} \partial_i b^\alpha M_{\alpha\beta} \partial_j b^\beta \\ &\quad - \frac{1}{2} e^{-4U} \gamma^{ij} \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right) \left(\partial_j \tilde{\psi} + \langle b, \partial_j b \rangle \right), \\ \frac{1}{\sqrt{\gamma}} \partial_i \left[\sqrt{\gamma} \gamma^{ij} G_{a\bar{a}} \partial_j \bar{z}^{\bar{a}} \right] &= \frac{\partial G_{b\bar{b}}}{\partial z^a} \gamma^{ij} \partial_i z^b \partial_j \bar{z}^{\bar{b}} + \frac{1}{2} e^{-2U} \gamma^{ij} \partial_i b^\alpha \frac{\partial M_{\alpha\beta}}{\partial z^a} \partial_j b^\beta. \end{aligned} \quad (2.10)$$

Within the class of the stationary metrics (2.2), we further select those with $D = 3$ Euclidean metric $\gamma_{ij} = \delta_{ij}$:

$$ds^2 = e^{2U} (dt + \omega_i dx^i)^2 - e^{-2U} dx^i dx^i. \quad (2.11)$$

Thus, we will henceforth restrict to consider $D = 4$ under-rotating BHs, with *flat* $D = 3$ base space. When replacing (2.2) with (2.11), Einstein equations (2.9) become a first order constraint:

$$G_{a\bar{a}} \partial_{(i} z^a \partial_{j)} \bar{z}^{\bar{a}} + \partial_i U \partial_j U + \frac{1}{2} e^{-2U} \partial_i b^\alpha M \partial_j b + \frac{1}{4} e^{-4U} \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right) \left(\partial_j \tilde{\psi} + \langle b, \partial_j b \rangle \right) = 0, \quad (2.12)$$

and the equations of motion (2.10) get slightly simplified:

$$\begin{aligned} \partial_i \left[e^{-4U} \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right) \right] &= 0, & (a) \\ \partial_i \left[e^{-2U} M_{\alpha\beta} \partial_i b^\beta \right] &= \partial_i \left[e^{-4U} \Omega_{\alpha\beta} b^\beta \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right) \right], & (b) \\ \partial_i \partial_i U &= -\frac{1}{2} e^{-2U} \partial_i b^\alpha M \partial_i b - \frac{1}{2} e^{-4U} \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right)^2, & (c) \\ \partial_i \left[G_{a\bar{a}} \partial_i \bar{z}^{\bar{a}} \right] &= \frac{\partial G_{b\bar{b}}}{\partial z^a} \partial_i z^b \partial_i \bar{z}^{\bar{b}} + \frac{1}{2} e^{-2U} \partial_i b^\alpha \frac{\partial M}{\partial z^a} \partial_i b. & (d) \end{aligned} \quad (2.13)$$

As a notation, let us introduce a $D = 3$ vector χ_i , which, by virtue of (2.13a), is divergenceless:

$$\chi_i \equiv e^{-4U} \left(\partial_i \tilde{\psi} + \langle b, \partial_i b \rangle \right), \quad \partial_i \chi_i = 0. \quad (2.14)$$

²Eq. (2.4a) is just the upper half of Eq. (2.8a).

Maxwell equations (2.13b) can immediately be integrated, yielding:

$$\partial_i b^\alpha = e^{2U} M^{\alpha\beta} \Omega_{\beta\gamma} \hat{H}_i^\gamma, \quad \hat{H}_i^\alpha \equiv H_i^\alpha + b^\alpha \chi_i, \quad \partial_i H_i^\alpha = 0, \quad (2.15)$$

where another $D = 3$ divergenceless vector H_i^α has been defined. The electromagnetic two-form field strength f_{ij}^α and the two-form W_{ij} corresponding to rotation are then just Hodge duals of the vectors χ_i and \hat{H}_i^α defined in (2.14)-(2.15):

$$W_{ij} = -\varepsilon_{ijk} \chi_k, \quad f_{ij}^\alpha = -\varepsilon_{ijk} \hat{H}_k^\alpha - b^\alpha W_{ij} = -\varepsilon_{ijk} H_k^\alpha.$$

This allows one to rewrite the Einstein equations (2.12) as well as the rest of the equations of motion (2.13) as follows:

$$G_{a\bar{a}} \partial_{(i} z^a \partial_{j)} \bar{z}^{\bar{a}} + \partial_i U \partial_j U - \hat{V}_{ij} e^{2U} + \frac{1}{4} e^{4U} \chi_i \chi_j = 0, \quad (2.16)$$

$$\begin{aligned} \partial_i \partial_i U &= e^{2U} \hat{V}_{ii} - \frac{1}{2} e^{4U} \chi_i^2, \\ \partial_i (G_{a\bar{a}} \partial_i \bar{z}^{\bar{a}}) &= \partial_a G_{b\bar{b}} \partial_i z^b \partial_i \bar{z}^{\bar{b}} + e^{2U} \partial_a \hat{V}_{ii}. \end{aligned} \quad (2.17)$$

In (2.16)-(2.17) the $D = 3$ Cartesian tensor

$$\hat{V}_{ij} \equiv -\frac{1}{2} \hat{H}_i^\top M \hat{H}_j \quad (2.18)$$

has been introduced; it will be referred to as *black hole potential*. In fact, it generalizes, for the whole class of metrics (2.11), the well known BH potential V_{BH} .

This latter was introduced in [18] for the case of a static and spherically symmetric BH. In our approach, this case corresponds to a vanishing χ_i and to the choice of the divergenceless vector H_i^α to be the gradient of an harmonic function:

$$\chi_i = 0, \quad H_i^\alpha = \partial_i H^\alpha. \quad (2.19)$$

In the case of spherical symmetry, H^α has a single pole where the black hole horizon resides (*i.e.* in the origin):

$$H^\alpha(x, y, z) = h^\alpha + P^\alpha \tau, \quad h^\alpha = \text{const}, \quad \tau = \frac{1}{\sqrt{x^2 + y^2 + z^2}}. \quad (2.20)$$

The constants P^α fit into the symplectic vector of electromagnetic charges (p^Λ, q_Λ) of the BH itself. Within these assumptions, the tensor \hat{V}_{ij} relates to V_{BH} as follows:

$$\hat{V}_{ij} = \partial_i \tau \partial_j \tau V_{BH}, \quad V_{BH} = -\frac{1}{2} P^\top M P. \quad (2.21)$$

Moreover, Maxwell Eqs. (2.15) decouple completely from (2.16) and (2.17), which get exactly the form presented in [18]. Therein, it was also shown that in the case in which the action (2.1) describes the bosonic sector of $N = 2$, $D = 4$ Maxwell-Einstein supergravity, the BH potential V_{BH} acquires a nice geometrical interpretation in terms of *special Kähler geometry* (see *e.g.* [19] for a review and a list of Refs.):

$$V_{BH} = Z \bar{Z} + G^{a\bar{a}} D_a Z \bar{D}_{\bar{a}} \bar{Z}, \quad (2.22)$$

where Z is the $N = 2$ central charge function and $D_a Z$ (matter charges) denotes its Kähler-covariant derivative:

$$Z = \langle P, V \rangle, \quad D_a Z = \langle P, D_a V \rangle, \quad (2.23)$$

defined in terms of the symplectic sections V^α of the flat symplectic bundle of special geometry [20, 21].

Interestingly, the Cartesian $D = 3$ tensor BH potential \hat{V}_{ij} (2.18) can be given a geometrical interpretation in the spirit of Eq. (2.22). Indeed, if one constructs a $D = 3$ *Cartesian generalization* of the central charge and of its covariant derivatives as

$$\hat{Z}_i \equiv \langle \hat{H}_i, V \rangle, \quad \hat{Z}_{ai} \equiv \langle \hat{H}_i, D_a V \rangle, \quad (2.24)$$

then it can be computed that

$$\hat{V}_{ij} = \hat{Z}_{(i} \hat{\bar{Z}}_{j)} + G^{a\bar{a}} \hat{Z}_{a(i} \hat{\bar{Z}}_{\bar{a}j)}, \quad (2.25)$$

which can be regarded as the generalization of (2.22) to generic, not necessarily axisymmetric, and thus possibly multi-centered, stationary solutions (at least those with flat three-dimensional spatial slices, *cfr.* (2.11)).

For later convenience, let us also here define the non-rotating limit ($\chi_i = 0$) of \hat{Z}_i and \hat{Z}_{ai} defined in (2.24):

$$Z_i = \langle H_i, V \rangle, \quad Z_{ai} = \langle H_i, D_a V \rangle. \quad (2.26)$$

As it will become evident from the subsequent treatment, both Cartesian $D = 3$ generalizations (2.24) and (2.26) of the $N = 2$ central charge Z and of its covariant derivatives $D_a Z$ will play an important role in the construction of first order equations of motion for multi-centered BHs.

3 First Order Equation of Motion

The first order equations of motion arise from the following *Ansatz*:

$$\partial_i U = e^U W_i, \quad \partial_i z^a = e^U \Pi_i^a, \quad \chi_i = 2e^{-U} \ell_i, \quad (3.1)$$

where W_i , Π_i^a and ℓ_i are functions of the scalar fields and, eventually, of some auxiliary ones. The choice of the *Ansatz* (3.1) is motivated by the fact that for single- and multi-centered BPS, as well for single-centered non-BPS BHs, the first order equations are of the form (3.1).

In the following treatment, we will conveniently consider special Kähler geometry tensors whose indices are “flattened” as usual:

$$V_{\underline{a}} \equiv E_{\underline{a}}^a D_a V, \quad \hat{Z}_{\underline{a}i} \equiv E_{\underline{a}}^a \hat{Z}_{ai} \quad \text{etc.}, \quad (3.2)$$

by using the *Vielbein* of the scalar manifold (for further details, see App. A). By means of (3.2), the first order equations (3.1) can be recast as follows:

$$\partial_i U = e^U W_i, \quad \partial_i z^a = e^U \Pi_i^a E_{\underline{a}}^a, \quad \chi_i = 2e^{-U} \ell_i. \quad (3.3)$$

Due to the Einstein equations (2.16), the functions entering the right hand sides of Eqs. (3.3) must satisfy the algebraic constraint

$$\hat{V}_{ij} = W_i W_j + \delta_{\underline{a}\underline{b}} \Pi_{(i}^{\underline{a}} \bar{\Pi}_{j)}^{\underline{b}} + \ell_i \ell_j, \quad (3.4)$$

expressing the relation between \hat{V}_{ij} and the $D = 3$ Cartesian fake superpotential W_i , and generalizing the relation [10]

$$V_{BH} = W^2 + 4G^{a\bar{a}}\partial_a W \bar{\partial}_{\bar{a}} W, \quad (3.5)$$

to which it reduces in the limit³ (2.19)-(2.21).

For later convenience, let us define the *complex* $D = 3$ Cartesian vector

$$\mathcal{W}_i \equiv W_i + i\ell_i, \quad (3.6)$$

in terms of which the algebraic constraint (3.4) can be recast as⁴ (*cfr.* (2.25))

$$\hat{V}_{ij} = \mathcal{W}_{(i} \overline{\mathcal{W}}_{j)} + \delta_{\underline{a}\underline{\bar{a}}} \Pi_{(i}^{\underline{a}} \overline{\Pi}_{j)}^{\underline{\bar{a}}}. \quad (3.7)$$

We anticipate that the complex vector \mathcal{W}_i will play an important role in integrating Maxwell Eqs. (2.15).

From now on, in order to avoid overloading the formulae, we omit underlining the flat indices.

3.1 Construction of Flows

We are now going to determine the first order flow-governing functions W_i , ℓ_i (or \mathcal{W}_i) and Π_i^a .

A crucial step in constructing such flow-governing functions is to use their expression in the BPS case, and then perform a suitable *flipping* of some (linear combinations of) electromagnetic charges.

3.1.1 BPS

As it is known [10, 1], in the *BPS class* [8], at spatial infinity (*i.e.*, where one may think of a restoration of the spherical symmetry), the first order superpotential W is just a combination of the ADM mass M and the NUT charge N , while the functions Π^a are the scalar charges⁵

$$W = \text{Re}(M + iN), \quad \Pi^a = \pi^a. \quad (3.8)$$

In turn, at spatial infinity it holds

$$M + iN = Z, \quad \pi^a = \overline{Z}_{\bar{a}}, \quad (3.9)$$

and thus (3.8) can be rewritten as:

$$W = \text{Re } Z, \quad \Pi^a = \overline{Z}_{\bar{a}}. \quad (3.10)$$

In order to proceed further, we will now specialize our treatment to the $N = 2$, $D = 4$ *stu* model [16], which can also be regarded as a common sector of $D = 4$ supergravity theories with rank-3 symmetric scalar manifolds.

Within this model, in order to restore manifest duality covariance in (3.10), one has to consider the proper action of the compact symmetry $[U(1)]^4 = H_4 \times U(1)$, where H_4 is the

³Within the same limit, the first order constraint (2.12) reduces to the Hamiltonian constraint given by Eq. (11) (with $c = 0$) of [18].

⁴For simplicity's sake, starting from below (3.7) we will refrain from underlining the scalar flat indices. In presence of curved indices, we hope the distinction will be clear from the context.

⁵Here, we omit the spatial index i because in the case under consideration the spherical symmetry reduces the number of independent components of a Cartesian vector to one.

stabilizer of the completely factorized $D = 4$ scalar manifold $[SL(2, \mathbb{R})/U(1)]^3$, or equivalently the maximal compact subgroup (*mcs*) of the $D = 4$ generalized electric-magnetic (U -)duality group $G_4 = [SL(2, \mathbb{R})]^3$, while the commuting $U(1)$ is the *mcs* of the $SL(2, \mathbb{R})$ Ehlers symmetry determined by the reduction to tree dimensions. As discussed in [1], under the resulting $[U(1)]^4$ symmetry, the central charge and its derivatives transform as follows⁶

$$Z \rightarrow e^{-\frac{i}{2}(\alpha_0 - \sum_a \alpha_a)} Z, \quad Z_a \rightarrow e^{-\frac{i}{2}(\alpha_0 - \alpha_a + \sum_{b \neq a} \alpha_b)} Z_a, \quad (3.11)$$

while the ADM mass M , NUT charge N and scalar charges π^a transform as

$$M + iN \rightarrow e^{\frac{i}{2}(\alpha_0 + \sum_a \alpha_a)} (M + iN), \quad \pi^a \rightarrow e^{-\frac{i}{2}(\alpha_0 + \alpha_a - \sum_{b \neq a} \alpha_b)} \pi^a. \quad (3.12)$$

Therefore, in order to restore manifest covariance in Eq. (3.10), one applies transformations (3.11) on the left and right hand sides of Eq. (3.9):

$$e^{\frac{i}{2}(\alpha_0 + \sum_a \alpha_a)} (M + iN) = e^{-\frac{i}{2}(\alpha_0 - \sum_a \alpha_a)} Z, \quad e^{-\frac{i}{2}(\alpha_0 + \alpha_a - \sum_{b \neq a} \alpha_b)} \pi^a = e^{\frac{i}{2}(\alpha_0 - \alpha_a + \sum_{b \neq a} \alpha_b)} \bar{Z}_{\bar{a}}; \quad (3.13)$$

by some trivial algebra, and taking into account Eq. (3.8), one achieves the following result, depending only on the phase α_0 :

$$W = \text{Re} \left(e^{-i\alpha_0} Z \right), \quad \Pi^a = e^{i\alpha_0} \bar{Z}_{\bar{a}}. \quad (3.14)$$

Although Eqs. (3.14) were obtained in [1] at spatial infinity only, we will postulate that they are valid not only at the spatial infinity, but *all along the corresponding whole scalar flow* (with generally broken spherical symmetry), and we will study the consequences of this approach in the next Sections. Our approach then justifies the following “*Cartesian generalization*” of (3.14):

$$W_i = \text{Re} \left(e^{-i\alpha_0} \hat{Z}_i \right), \quad \Pi_i^a = e^{i\alpha_0} \hat{\bar{Z}}_{\bar{a}i}, \quad (3.15)$$

with \hat{Z}_i and \hat{Z}_{ai} defined in (2.24). Consequently, the function ℓ_i , and hence χ_i , is easily deduced from the Einstein constraint (3.7):

$$\ell_i = \text{Im} \left(e^{-i\alpha_0} \hat{Z}_i \right) \quad \Rightarrow \quad \chi_i = 2e^{-U} \text{Im} \left(e^{-i\alpha_0} \hat{Z}_i \right). \quad (3.16)$$

Therefore, one realizes that the complex vector $M + iN$ (composed by the ADM mass M and the NUT charge N) can be “prolonged” all along the scalar flow by the Cartesian $D = 3$ vector \mathcal{W}_i (3.6), which in this case reads

$$\mathcal{W}_i = W_i + i\ell_i = e^{-i\alpha_0} \hat{Z}_i. \quad (3.17)$$

The phase α_0 becomes a dynamical field and the consistency of the first order equations require that it satisfy the following equation

$$\partial_i \alpha_0 = e^U \text{Im} (e^{-i\alpha_0} \hat{Z}_i) - \text{Im} (\partial_a K \partial_i z^a).$$

⁶The phases α_0 and α_a ($a = 1, 2, 3$) are not directly related to the Ehlers $U(1)$ and to the three $U(1)$ ’s in H_4 , but rather they are a linear combinations of the corresponding phases thereof; for further detail, see [1].

It is here worth stressing that Eqs. (3.16) resemble the ones obtained in [8], but there is an important difference: they are indeed expressed in terms of \hat{Z}_i , instead of Z_i . The very definition (2.24) of the $D = 3$ Cartesian $N = 2$ central charge \hat{Z}_i involves terms proportional to the vector χ_i , thus Eq. (3.16) can actually be solved for χ_i itself, achieving the following expression:

$$\chi_i = \frac{2e^{-U}\text{Im}[e^{-i\alpha_0}Z_i]}{1 - 2e^{-U}\text{Im}[e^{-i\alpha_0}\langle b, V \rangle]}. \quad (3.18)$$

Then, the integration of Maxwell Eqs. (2.15) exactly implies the result of [8]. In order to achieve this, one is hinted by the fact that expression (3.15) can be recast in the following form:

$$W_i = \text{Re} \langle \hat{H}_i, e^{-i\alpha_0} V \rangle,$$

and one can thus check that

$$b^\alpha = 2e^U \text{Re}(e^{-i\alpha_0} V^\alpha) \quad (3.19)$$

satisfies the Maxwell equations (2.15), thus yielding the following flow-defining functions:

$$W_i = \text{Re}(e^{-i\alpha_0} Z_i), \quad \Pi_i^a = e^{i\alpha_0} \bar{Z}_{\bar{a}i}, \quad \chi_i = -2e^{-U} \text{Im}(e^{-i\alpha_0} Z_i), \quad (3.20)$$

which govern the first-order formulation of scalar flows of the *BPS class* of multi-centered BHs [8].

3.1.2 Almost BPS

In order to construct a first order formalism for non-BPS flows, we will exploit suitable *charge flippings* at spatial infinity, and then consistently extend them all along the flow. The relevant flippings of charges have been derived in [1] by exploiting an analysis of the relevant nilpotent orbits of $SO(4, 4)$, which is the $D = 3$ duality group of the *stu* model.

We start and consider the central charge Z and its flat derivatives Z_a at spatial infinity. Without loss of generality (by $H_4 = U(1)^3$ duality), the asymptotical values of the scalar fields z^a can be set to

$$z^a = -i. \quad (3.21)$$

Therefore, at spatial infinity the central charge along with its flat Kähler-covariant derivatives respectively read

$$Z = \frac{1}{2\sqrt{2}} \left[q_0 + ip^0 - \sum_a (p^a - iq_a) \right], \quad Z_a = \frac{1}{2\sqrt{2}} \left[q_0 - ip^0 + \sum_{b \neq a} (p^b + iq_b) - p^a + iq_a \right]. \quad (3.22)$$

Instead of dealing with the charges (p^0, p^a, q_0, q_a) , let us define a basis (D^0, D^a, D_0, D_a) whose interpretation in terms of D -brane charges is given below [1]:

$$\begin{aligned} \text{D6} : D^0 &\equiv \frac{1}{2} \left[p^0 + \sum_a q_a \right], & \text{D4} : D^a &\equiv p^a, \\ \text{D2} : D_a &\equiv \frac{1}{2} \left[p^0 + q_a - \sum_{b \neq a} q_b \right], & \text{D0} : D_0 &\equiv q_0; \end{aligned} \quad (3.23)$$

this is a more convenient basis on which one can act with the charge flipping. In terms of D -charges (3.23), (3.22) can be rewritten as

$$\begin{aligned} Z &= \frac{1}{2\sqrt{2}} \left[D_0 - iD^0 + \sum_a (D^a + iD_a) \right], \\ Z_a &= \frac{1}{2\sqrt{2}} \left[D_0 - iD^0 - \sum_{b \neq a} (D^b + iD_b) + D^a + iD_a \right], \end{aligned} \quad (3.24)$$

or equivalently, inverting in terms of the D -charges:

$$\begin{aligned} D_0 &= \frac{1}{\sqrt{2}} \operatorname{Re} \left[Z + \sum_a Z_a \right], \quad D_a = \frac{1}{\sqrt{2}} \operatorname{Im} \left[Z + Z_a - \sum_{b \neq a} Z_b \right], \\ D^0 &= -\frac{1}{\sqrt{2}} \operatorname{Im} \left[Z + \sum_a Z_a \right], \quad D^a = \frac{1}{\sqrt{2}} \operatorname{Re} \left[Z + Z_a - \sum_{b \neq a} Z_b \right]. \end{aligned} \quad (3.25)$$

Now, consistent with the analysis of [1], let us flip the sign of the brane charge D_0 (3.23) in (3.24), obtaining

$$\begin{aligned} \tilde{Z} &\equiv Z|_{D_0 \rightarrow -D_0} = \frac{1}{2\sqrt{2}} \left[-D_0 - iD^0 + \sum_a (D^a + iD_a) \right], \\ \tilde{Z}_a &\equiv Z_a|_{D_0 \rightarrow -D_0} = \frac{1}{2\sqrt{2}} \left[-D_0 - iD^0 - \sum_{b \neq a} (D^b + iD_b) + D^a + iD_a \right]; \end{aligned} \quad (3.26)$$

by plugging the expressions (3.25) into (3.26), one achieves the following result:

$$\tilde{Z} = \frac{1}{4} \left[3Z - \bar{Z} - 2 \operatorname{Re} \sum_a Z_a \right], \quad \tilde{Z}_a = -\frac{1}{2} \left[\operatorname{Re} Z + \operatorname{Re} \sum_b Z_b - 2Z_a \right], \quad (3.27)$$

which can then be plugged into the right hand side of Eqs. (3.9). By restoring manifest $(U(1))^4$ covariance in the way described above, and extending these relations along the whole flow and not only at spatial infinity, one obtains the following expressions, explicitly depending on *all four* phases α_0 and α_a 's:

$$\begin{aligned} W_i &= \frac{1}{4} \operatorname{Re} \left[e^{-i\alpha_0} \left(3 - e^{i(\alpha_0 + \sum_a \alpha_a)} \right) \hat{Z}_i - \left(1 + e^{-i(\alpha_0 + \sum_a \alpha_a)} \right) \sum_a e^{i\alpha_a} \hat{Z}_{ai} \right]; \\ \Pi_i^a &= \frac{1}{2} e^{i\alpha_0} \hat{\bar{Z}}_{ai} - \frac{1}{2} e^{\frac{i}{2}(\alpha_0 + \alpha_a - \sum_{b \neq a} \alpha_b)} \operatorname{Re} \left(e^{-\frac{i}{2}(\alpha_0 - \sum_b \alpha_b)} \hat{Z}_i \right) \\ &\quad - \frac{i}{2} e^{\frac{i}{2}(\alpha_0 + \alpha_a - \sum_{b \neq a} \alpha_b)} \operatorname{Im} \left[\sum_b e^{-\frac{i}{2}(\alpha_0 - \alpha_b + \sum_{c \neq b} \alpha_c)} \hat{Z}_{bi} \right]. \end{aligned} \quad (3.28)$$

This case corresponds to the so-called *almost BPS* class [3, 1], whose first order formulation will be studied in Sec. 5.

3.1.3 Composite Non-BPS

Another possibility is to flip not only D_0 , but also D_a brane charges (3.23) as well, namely:

$$D_0 \rightarrow -D_0, \quad D_a \rightarrow -D_a. \quad (3.29)$$

By applying the very same procedure described above for the sign flip (3.29), one finally gets the following expressions, explicitly depending on only *three* phases α_a 's:

$$\begin{aligned} W_i &= \frac{1}{2} \text{Re} \left(e^{i \sum_a \alpha_a} \hat{Z}_i - \sum_a e^{i \alpha_a} \hat{Z}_{ai} \right); \\ \Pi_i^a &= -\frac{1}{2} \left[e^{i \alpha_a} \hat{Z}_i - e^{i(\alpha_a - \sum_{b \neq a} \alpha_b)} \hat{Z}_{ai} + \sum_{b \neq a} e^{-i \alpha_b} \hat{Z}_{bi} \right]. \end{aligned} \quad (3.30)$$

This case corresponds to the so-called *composite non-BPS* class [1], whose first order formulation will be studied in Sec. 4.

4 Composite Non-BPS Class

We will start analyzing the *composite non-BPS* [1] class (3.30), its structure being relatively simpler than the one of the almost-BPS class.

In the previous Section, we constructed the functions W_i and Π_i^a (3.30), governing the first order flow Eqs. (3.1). From Einstein Eqs. (2.16), one can deduce the following expression for the remaining flow-governing function, namely the $D = 3$ Cartesian rotation vector ℓ_i :

$$\ell_i = -\frac{1}{2} \text{Im} \left[e^{i \sum_a \alpha_a} \hat{Z}_i - \sum_a e^{i \alpha_a} \hat{Z}_{ai} \right]. \quad (4.1)$$

As it can be realized by looking at previous equations, the various α -phases play an important role in the first order formalism for the multi-centered scalar flows. Nevertheless, their own dynamics has not been concerned so far. In the case This can be uniquely determined if one imposes that after differentiation of the first order equations (3.3), (3.30) and (4.1), the second order ones (2.17) are obtained. This requirement yields the following three partial differential Eqs. for the three phases α_a 's ($a = 1, 2, 3$) in the composite non-BPS multi-centered class⁷:

$$\begin{aligned} \partial_i \alpha_a &= \frac{1}{2} e^U \text{Im} \left[\left(e^{i \sum_b \alpha_b} + e^{i \alpha_a} - \sum_{b \neq a} e^{i \alpha_b} \right) \hat{Z}_i + 2e^{i \alpha_a} \hat{Z}_{ai} - 2e^{-i \alpha_a} \sum_{b \neq a} \hat{Z}_{bi} \right. \\ &\quad \left. + e^{-i \alpha_a} \left(e^{-i \sum_b \alpha_b} - e^{i \alpha_a} + \sum_{b \neq a} e^{-i \alpha_b} \right) \left(-e^{i \alpha_a} \hat{Z}_{ai} + \sum_{b \neq a} e^{i \alpha_b} \hat{Z}_{bi} \right) \right]. \end{aligned} \quad (4.2)$$

Thus, one can conclude that Eqs. (3.3), (3.30), (4.1), and (4.2) – together with Maxwell Eqs. (2.15) – constitute the *first order formalism for composite non-BPS multi-centered BH solutions* in $N = 2$, $D = 4$ *stu* model.

⁷It is worth noticing that both sets (4.2) and (5.2) of partial differential Eqs. for the phases (respectively in the composite non-BPS e almost BPS classes) exhibit a manifest *stu triality* symmetry [16].

Let us now integrate further Maxwell Eqs. (2.15) by exploiting the hint described in Sec. 3.1.1 for the BPS flow. Namely, one can easily notice that for the composite non-BPS class the complex $D = 3$ Cartesian vector \mathcal{W}_i (3.6) can be rewritten as

$$\mathcal{W}_i = \langle \hat{H}_i, T \rangle, \quad (4.3)$$

where the complex symplectic vector

$$T^\alpha \equiv \frac{1}{2} \left(e^{-i \sum_a \alpha_a} \bar{V}^\alpha - \sum_a e^{-i \alpha_a} \bar{V}_a^\alpha \right) \quad (4.4)$$

was defined. After a little algebra, one also finds that electromagnetic potential b (2.6) enjoys an expression similar to the one holding for the BPS class (3.19), namely:

$$b^\alpha = 2 e^U \text{Re} [T^\alpha (1 - i B e^{2U})], \quad (4.5)$$

for some function B , which can be considered as the $D = 5$ *rotational contribution* to the electromagnetic potential.

As it occurred in the BPS multi-centered class treated in Sec. 3.1.1, also in the composite non-BPS class Eq. (4.1) contains χ_i in both sides. Remarkably, in this class the dependence on χ_i drops out when replacing (4.5) into (4.1); as a consequence, Eq. (4.1) turns into the following algebraic constraint:

$$\text{Im} \left(e^{i \sum_a \alpha_a} Z_i - \sum_a e^{i \alpha_a} Z_{ai} \right) = 0, \quad (4.6)$$

and the $D = 3$ Cartesian rotation vector χ_i can be computed to read

$$\chi_i = \partial_i B + B e^U \text{Re} \left(3 e^{i \sum_a \alpha_a} Z_i - \sum_a e^{i \alpha_a} Z_{ai} \right) - 2 e^{-U} \text{Im} \left(e^{i \sum_a \alpha_a} Z_i \right), \quad (4.7)$$

where the function B occurred for the first time in (4.5), and it should be such that χ_i is divergenceless (*cfr.* (2.14)).

The counting of the two “flat” directions along the (non-BPS, “large”) flows in the *stu* model [12, 23] is retrieved by considering that in this class the tree partial differential Eqs. (4.2) for the tree phases α_a ’s are supplemented by the algebraic constraint (4.6) (see also the treatment of next Subsection).

4.1 Single-Centered Solutions

Consistent with [1] (see also *e.g.* the two-centered analysis in [22]), we will now show that the well known case of a single-centered non-rotating non-BPS BH can be obtained by performing a suitable limit in the multi-centered class under consideration.

We consider the single-centered limit (2.19)-(2.20), in which the divergenceless vector H_i^α introduced in (2.15) is realized as a derivative of an harmonic function H^α defined in (2.20):

$$H_i^\alpha = \partial_i H^\alpha, \quad \partial_i \partial_i H^\alpha = 0. \quad (4.8)$$

Before proceeding further, it is worth clarifying here the meaning of the constraint (4.6). With such a purpose, let us introduce the three real values

$$\lambda^a \equiv \frac{\text{Im}(e^{i \beta_a} z^a)}{\text{Im}(e^{i \beta_a})}, \quad \beta_a \equiv \frac{1}{2} \sum_{b \neq a} \alpha_b \quad (4.9)$$

and construct an *stu*-like symplectic section

$$V_\lambda^\alpha \equiv (1, \lambda^1, \lambda^2, \lambda^3, -\lambda^1 \lambda^2 \lambda^3, \lambda^2 \lambda^3, \lambda^1 \lambda^3, \lambda^1 \lambda^2)^T. \quad (4.10)$$

This allows the following rather elegant rewriting of the constraint (4.6):

$$\langle P, V_\lambda \rangle = 0. \quad (4.11)$$

Note that the three λ^a 's satisfy the single constraint (4.11), and therefore only two of them are independent; this reflects the existence of two “flat” directions along the non-rotating single-centered *stu* non-BPS flow [12, 23]. A noteworthy feature of λ^a 's is that they are in fact *integrals of motion* in $D = 3$ flat space; indeed, from the above Eqs. one can check that

$$\partial_i \lambda^a = 0. \quad (4.12)$$

By inverting the definitions (4.9), one can thus express the three phases α_a 's in terms of the scalar fields z^a and of the integrals of motion λ^a 's themselves:

$$e^{i \sum_{b \neq a} \alpha_b} = \frac{\lambda^a - \bar{z}^{\bar{a}}}{\lambda^a - z^a}. \quad (4.13)$$

In the case of spherical symmetry, each of the flow-defining $D = 3$ Cartesian vectors W_i and Π_i^a can actually be reduced to only one independent function:

$$W_i = \partial_i \tau W, \quad \Pi_i^a = \partial_i \tau \Pi^a, \quad (4.14)$$

where τ has been defined in (2.20); therefore, the spatial vector index i can formally be neglected.

By plugging the phases (4.13) into the Eqs. (3.30), one retrieves the known expression for the non-BPS fake superpotential W_{nBPS} for a non-rotating single-centered non-BPS BH in *stu* model (with non-vanishing Z at the horizon) [12]; in order to achieve a complete agreement in notations, one should parametrize the integrals of motion as

$$\lambda^a \equiv \frac{e^{\alpha_a} \nu \xi_a + \rho_a}{e^{\alpha_a} \nu - 1}, \quad (4.15)$$

where the α_a 's are some constant (not to be confused with the phases α_a 's) satisfying $\sum_a \alpha_a = 0$.

It should be here remarked that in the non-rotating ($\chi_i = 0$) single-centered non-BPS BH solution supported by the Kaluza-Klein (p^0, q_0) -configuration [24, 12]

$$e^{-4U} = H_0 H_1 H_2 H_3 - B^2, \quad z^a = \frac{A_a^- - 2ie^{-2U}}{A_a^+ - 2B}; \quad (4.16)$$

$$B = \text{const}, \quad A_a^\pm = \frac{H_1 H_2 H_3}{H_a} \pm H_0 H_a, \quad (4.17)$$

the function χ_i , defined by eq. (4.7), vanishes. On the other hand, if the function B is not a constant, then it turns out to read

$$B = b + J \frac{\cos \theta}{r^2},$$

where J is an angular momentum; this corresponds to the Rasheed-Larsen rotating ($\chi_i \neq 0$) single-centered non-BPS BH [25, 26].

4.2 Multi-Centered Solutions

4.2.1 Electric Configuration

In [1], it was found that for the composite non-BPS class there exists a particular solution

$$e^{-4U} = 4 V_0 H_1 H_2 H_3 - B^2, \quad z^a = \frac{1}{2} \frac{2K_a \frac{H_1 H_2 H_3}{H_a} - B - i e^{-2U}}{K_a^2 \frac{H_1 H_2 H_3}{H_a} + H_a V_0 - B K_a}, \quad (4.18)$$

defined in terms of functions V_0 , H_a and K_a satisfying the following equations:

$$\begin{aligned} \partial_i V_0 &= H_i^0 - K_a H_i^a + \frac{1}{2} |\epsilon^{abc}| K_a K_b H_{ci}, \\ \partial_i K_a &= \frac{H_a}{2 H_1 H_2 H_3} \left[\sum_{b \neq a} H_a H_i^a - H_b H_i^b + |\epsilon^{bcd}| K_b H_c H_{di} - 2 H_a \sum_{\substack{b \neq a \neq c \\ b \neq c}} K_b H_{ci} \right], \\ \chi_i &= \partial_i B - |\epsilon^{abc}| \partial_i K_a H_b H_c, \end{aligned} \quad (4.19)$$

in which H_i^0 , H_i^a and $H_{ai} = \partial_i H_a$ are components of the divergenceless symplectic vector H_i^α introduced in (2.15). Without entering in the details *e.g.* of the two-centered solution, it should be remarked here that in the composite non-BPS class the fact that H_i^0 and H_i^a are not harmonic implies the centers to exhibit *mutually non-local* electric-magnetic fluxes. The same holds for the magnetic solution considered in Sec. 4.2.2.

One can check that this solution satisfies the composite non-BPS first order equations derived above if the phases α_a 's are fixed as follows:

$$\frac{1}{2} \sum_{b \neq a} \alpha_b = \pi - \arg z^a. \quad (4.20)$$

The condition (4.20) is a particular solution to the partial differential equations (4.2), and it actually fixes the two non-BPS “flat” directions, whose presence cannot thus be recognized in the explicit form (4.18)-(4.19) of this solution. Moreover, it should be noted that (4.20) yields the vanishing of the remaining components (not entering (4.19)) of the divergenceless symplectic vector H_i^α :

$$H_{0i} = 0. \quad (4.21)$$

Physically, this means that the graviphoton electric charge q_0 pertaining to *each* BH center should vanish. However, in the two-centered case, it can be checked that in general the charge vectors pertaining to each of the two centers are *mutually non-local*, even if there are no solutions in this class in which all two-centered duality invariants [27] are independent (also *cfr.* the analysis in [22]).

Multi-Centered Integrals of Motion Although introduced for single-centered BH solutions, the definition (4.9) of integrals of the $D = 3$ motion still holds for multi-centered BHs. In this case, the constraint (4.6) can be rewritten as

$$\left[\prod_a \text{Im } e^{i\beta_a} \right] \langle H_i, V_\lambda \rangle = 0. \quad (4.22)$$

Assuming that the phases α_a are such that the product of the imaginary parts (recall definition (4.9)) does not vanish, one obtains that the skew-symmetric symplectic product $\langle H_i, V_\lambda \rangle$ necessarily vanishes:

$$\langle H_i, V_\lambda \rangle = H_{0i} + H_{ai}\lambda^a - 3H_i^{(1}\lambda^2\lambda^3) + H_i^0\lambda^1\lambda^2\lambda^3 = 0. \quad (4.23)$$

From the definition of integrals of the $D = 3$ motion, and especially from its inverse (4.13), one can see that (4.20) holds *iff* all integrals of motion equal to zero: $\lambda^a = 0, \forall a = 1, 2, 3$. In order to fulfill the constraint (4.23), one has to require $H_{0i} = 0$, matching (4.21).

The product of the imaginary parts in (4.22) might vanish. This happens for example in the case of the so-called magnetic configuration, which analysis is performed below.

4.2.2 Magnetic Configuration

If all phases α_a vanish, then the constraint (4.2) turns into an identity. Despite the fact that the definition (4.9) of integrals of motion becomes singular, a careful analysis yields that the constraint (4.22) is satisfied when

$$H_i^0 = 0. \quad (4.24)$$

Physically, this condition means that the graviphoton magnetic charge p^0 of *each* BH center should vanish. This case was considered in [2]; therein, an explicit solution in the magnetic configuration

$$e^{-4U} = 4V_0 H^1 H^2 H^3 - B^2, \quad z^a = K^a + \frac{H^a}{2H^1 H^2 H^3} (B - ie^{-2U}) \quad (4.25)$$

was found, in which the function V_0 as well as the harmonic functions K^a satisfy the first order equations (no summation over index a):

$$\begin{aligned} \partial_i V_0 &= -H_{0i} - K^b H_{bi} + \frac{1}{2} |\epsilon_{bcd}| K^b K^c \partial_i H^d, \\ \partial_i K^a &= \frac{H^a}{2H^1 H^2 H^3} \left[H^a H_{ai} - \sum_{b \neq a} H^b H_{bi} + |\epsilon_{bcd}| H^b K^c \partial_i H^d - 2H^a \sum_{\substack{b \neq a \neq c \\ b \neq c}} K^b H_i^c \right]. \end{aligned} \quad (4.26)$$

Thus, the $D = 3$ Cartesian rotation vector reads

$$\chi_i = \partial_i B + |\epsilon_{abc}| \partial_i K^a H^b H^c. \quad (4.27)$$

This solution, which is a *particular* solution of the composite non-BPS first order equations, is written in terms of the functions H_{0i} , H_{ai} and $H_i^a = \partial_i H^a$, which are components of the divergenceless symplectic vector H_i^α introduced in (2.15).

5 Almost BPS Class

For the *almost BPS* [3, 1] class of flows, the first order equations are given by the general Eqs. (3.3) supplemented with the flow-defining functions W_i and Π_i^a given by Eqs. (3.28). Then, one has to satisfy the Einstein constraint (3.7), yielding the following expression for the $D = 3$ Cartesian rotation vector ℓ_i :

$$\ell_i = \frac{1}{4} \text{Im} \left[e^{-i\alpha_0} \left(3 + e^{i(\alpha_0 + \sum_a \alpha_a)} \right) \hat{Z}_i + \left(1 - e^{-i(\alpha_0 + \sum_a \alpha_a)} \right) \sum_a e^{i\alpha_a} \hat{Z}_{ai} \right]. \quad (5.1)$$

As described in the Sec. 4, if one requires the equivalence of the first order equations to their second order ancestors, the following four partial differential Eqs. for the four phases α_0 and α_a 's are obtained:

$$\begin{aligned} \partial_i \alpha_0 = & e^U \text{Im} \left[e^{-i\alpha_0} (\hat{Z}_i + \sum_a \hat{Z}_{ai}) \right] \\ & - \frac{1}{4} e^U \left[\left(e^{i \sum_a \alpha_a} - \sum_a e^{i\alpha_a} + e^{-i\alpha_0} \left(\sum_a e^{i \sum_{b \neq a} \alpha_b} - 1 \right) \right) \hat{Z}_i \right. \\ & \left. + \sum_a \left[(e^{i\alpha_a} + e^{-i\alpha_0}) (1 - e^{-i \sum_{b \neq a} \alpha_b}) - \sum_{b \neq a} e^{-i\alpha_b} (1 - e^{i\alpha_a - i\alpha_0}) \right] \hat{Z}_{ai} \right], \end{aligned} \quad (5.2)$$

$$\begin{aligned} \partial_i \alpha_a = & -\partial_i \alpha_0 - \frac{1}{2} e^U \cos \frac{(\alpha_0 + \alpha_a)}{2} \text{Im} \left[e^{-\frac{i}{2}(\alpha_0 + \alpha_a)} \left[\left(e^{i \sum_{b \neq a} \alpha_b} - 1 \right) \hat{Z}_i \right. \right. \\ & \left. \left. - \left(e^{-i \sum_{b \neq a} \alpha_b} + 3 \right) \hat{Z}_{ai} \right] + e^{-\frac{i}{2}(\alpha_0 + \alpha_a)} \sum_{b \neq a} (e^{i\alpha_b} - e^{-i\alpha_c}) \hat{Z}_{bi} \right], \end{aligned}$$

in which the index c must be different from both a and b . In turn, Eqs. (5.2) yield a useful relation:

$$\partial_i \left[\alpha_0 + \sum_a \alpha_a \right] = 2 e^U \text{Re} \left[e^{\frac{i}{2}(\alpha_0 + \sum_a \alpha_a)} \right] \text{Im} \left[e^{\frac{i}{2}(\alpha_0 + \sum_a \alpha_a)} \left(\sum_a e^{i\alpha_a} \hat{Z}_{ai} - e^{-i(\alpha_0 + \sum_a \alpha_a)} \hat{Z}_i \right) \right]. \quad (5.3)$$

In order to integrate the Maxwell Eqs. (2.15), we exploit the same approach considered in Sec. 4; the flow-defining functions W_i and χ_i , as well as the electromagnetic potential b^α , can again be represented as in (4.3) and (4.5) in terms of symplectic vector T , which for the class under consideration has a slightly more complicated expression, namely:

$$T^\alpha = -\frac{1}{4} \left(e^{-i \sum_a \alpha_a} \bar{V}^\alpha - 3e^{-i\alpha_0} V^\alpha + \sum_a e^{-i\alpha_a} \bar{V}_a^\alpha + e^{-i(\alpha_0 + \sum_a \alpha_a)} \sum_a e^{i\alpha_a} V_a^\alpha \right). \quad (5.4)$$

When replacing the electromagnetic potential b expressed in terms of T (5.4) into Eq. (5.1), this latter is turned into a constraint:

$$\text{Im} \left[e^{-i\alpha_0} \left(3 + e^{i(\alpha_0 + \sum_a \alpha_a)} \right) Z_i + \left(1 - e^{-i(\alpha_0 + \sum_a \alpha_a)} \right) \sum_a e^{i\alpha_a} Z_{ai} \right] = 0. \quad (5.5)$$

In other words, as for the composite non-BPS class treated in Sec. 4, if one tries to deduce χ_i from (5.1) recalling that (*cfr.* (2.15) and (2.24))

$$\hat{Z}_i = Z_i + \chi_i \langle b, V \rangle, \quad \hat{Z}_{ai} = Z_{ai} + \chi_i \langle b, V_a \rangle, \quad (5.6)$$

then it can be checked that the dependence of (5.1) on χ_i drops out, thus yielding the constraint (5.5).

However, differently from the composite non-BPS class, in the almost BPS class, the Maxwell equations turn out to be consistent *only when two further constraints are satisfied*, namely:

$$\text{I} : \alpha_0 + \sum_a \alpha_a = \pi \quad \text{or} \quad \text{II} : \alpha_0 + \sum_a \alpha_a = 2 \arctan(e^{2U} B), \quad (5.7)$$

thus giving rise, as in the t^3 model investigated in [14], to two distinct almost BPS branches⁸, which we will separately analyze in the next two Subsections.

The counting of the two “flat” directions along the (non-BPS, “large”) flows in the stu model [12, 23] is retrieved by considering that in this class the four partial differential Eqs. (5.2) for the four phases α_0 and α_a ’s are supplemented by the algebraic constraint (5.5) and by the further condition **I** or **II** of (5.7).

5.1 Branch I

In the case **I** of (5.7), the $D = 3$ Cartesian rotation vector reads

$$\chi_i = \partial_i B + e^U B \operatorname{Re} \left[\sum_a e^{i\alpha_a} Z_{ai} + 3e^{-i\alpha_0} Z_i \right] + e^{-U} \operatorname{Im} \left[\sum_a e^{i\alpha_a} Z_{ai} - e^{-i\alpha_0} Z_i \right], \quad (5.8)$$

and expressions for the flow-defining functions gets slightly simplified as follows:

$$\begin{aligned} W_i &= \operatorname{Re} \left[e^{-i\alpha_0} \hat{Z}_i \right], \\ \Pi_i^a &= \frac{1}{2} e^{i\alpha_0} \hat{Z}_{ai} - \frac{i}{2} e^{i(\alpha_0 + \alpha_a)} \operatorname{Im} \left[e^{-i\alpha_0} \hat{Z}_i \right] + \frac{1}{2} e^{i(\alpha_0 + \alpha_a)} \operatorname{Re} \left[\sum_b e^{i\alpha_b} \hat{Z}_{bi} \right]. \end{aligned} \quad (5.9)$$

This branch contains BPS multi-centered solutions [8] with a particular constraint. This can be proved by simply setting $B = 0$ in (5.8), thus implying the functions W_i and Π_i^a to turn into their BPS counterparts (3.15), with rotation χ_i acquiring the following form:

$$\chi_i = e^{-U} \operatorname{Im} \left[e^{i \sum_a \alpha_a} Z_i + \sum_a e^{i\alpha_a} Z_{ia} \right]. \quad (5.10)$$

Independently on the value of the B -field, the constraint (5.5) takes the form

$$\operatorname{Im} \left[e^{i \sum_a \alpha_a} Z_i - \sum_a e^{i\alpha_a} Z_{ia} \right] = 0. \quad (5.11)$$

Thus, (5.10) and (5.11) yield

$$\chi_i = -2 e^{-U} \operatorname{Im} \left[e^{-i\alpha_0} Z_i \right] = 2 e^{-U} \operatorname{Im} \left[\sum_a e^{i\alpha_a} Z_{ai} \right], \quad (5.12)$$

which is exactly the expression of the $D = 3$ rotation pertaining to the BPS multi-centered class [8] subjected to the additional constraint (5.11).

The actual meaning of the constraint (5.11) has yet to be clarified; nevertheless, we can here make some observations. Considering the corresponding single-centered case, and thus setting both B and χ_i to zero, this constraint splits in two parts, namely:

$$\operatorname{Im} \left[\sum_a e^{i\alpha_a} Z_{ai} \right] = 0 \quad \text{and} \quad \operatorname{Im} \left[e^{i \sum_a \alpha_a} Z_i \right] = 0. \quad (5.13)$$

⁸It is interesting to notice that each branch admits a single-centered limit of different type: while branch **I** admits a particular BPS BH with $i_3 = 0$ (*cfr.* Sec. 5.1), branch **II** exhibits a non-BPS BH (as single-centered limit of the particular set of embedded composite non-BPS solutions discussed in Sec. 5.2). This is consistent with the fact that two-centered almost BPS solutions are characterized by a BPS and a non-BPS center [1, 22].

In this case, *at least* a particular solution

$$\alpha_a = -\arg Z_a, \quad \arg Z = \sum_a \arg Z_a \quad (5.14)$$

of the constraint **I** of (5.7) can be shown to be consistent with the first order equations and, as a consequence, the duality invariant i_3 [28, 13] vanishes. In this sense, the constraint (5.11) can be regarded as the generalization, within the BPS multi-centered flow embedded into the branch **I** of the almost BPS class, of the constraint $i_3 = 0$.

At the moment, it is not known whether this branch contains other consistent multi-centered solutions different from the aforementioned constrained BPS solutions; we leave this issue for further investigation.

5.2 Branch II

In the case **II** of (5.7), the $D = 3$ Cartesian rotation vector reads

$$\chi_i = \partial_i B - B e^U \operatorname{Re} \left[3 e^{i \sum_a \alpha_a} Z_i - \sum_a e^{i \alpha_a} Z_{ai} \right] + e^{-U} \operatorname{Im} \left[3 e^{i \sum_a \alpha_a} Z_i - \sum_a e^{i \alpha_a} Z_{ai} \right], \quad (5.15)$$

and the constraint (5.5) acquires the following form:

$$\begin{aligned} 2 \operatorname{Im} \left[e^{i \sum_a \alpha_a} Z_i \right] - B e^{2U} \operatorname{Re} \left[3 e^{i \sum_a \alpha_a} Z_i - \sum_a e^{i \alpha_a} Z_{ai} \right] \\ - B^2 e^{4U} \operatorname{Im} \left[e^{i \sum_a \alpha_a} Z_i - \sum_a e^{i \alpha_a} Z_{ai} \right] = 0. \end{aligned} \quad (5.16)$$

Here we refrain from writing down the explicit expressions for the flow-defining functions W_i and Π_i^a , resulting from plugging Eq. (5.7) (case **II**) into (3.28), because in this branch their form is not very illuminating and rather cumbersome.

It is worth remarking that, among all possible solutions to Eqs. (5.15)-(5.16), there is a particular one that coincides with a particular solution of the composite non-BPS treated in Sec 5; indeed, if the B -field is set to a constant and the rotation χ_i vanishes, then it can be checked that the set of equations (5.15)-(5.16) turns into the set of Eqs. (4.6) and (4.7), which describes the composite non-BPS class.

An Example with Fixed “Flat” Directions An interesting example is provided by the choice of all phases α_a ’s to be equal. The consistency of Eqs. (5.2) within the case **II** of (5.7) requires then that

$$\alpha_1 = \alpha_2 = \alpha_3 = -\frac{\pi}{2} + \arctan B e^{2U}, \quad \alpha_0 = \frac{3\pi}{2} - \arctan B e^{2U}. \quad (5.17)$$

This latter exactly falls into the class of solutions [3, 1]

$$e^{-4U} = 4H^0 \mathcal{Z}_1 \mathcal{Z}_2 \mathcal{Z}_3 - B^2, \quad z^a = K^a - \frac{B - i e^{2U}}{2H^0 \mathcal{Z}_a}, \quad (5.18)$$

where H^0 is harmonic, and the functions \mathcal{Z}_a and K^a satisfy the following equations:

$$\partial_i \mathcal{Z}_a = H_{ia} - |\epsilon_{abc}| \left(H_i^b - \frac{1}{2} K^b \partial_i H^0 \right) K^c, \quad \partial_i K_a = -\frac{1}{H^0} (H_i^a - K^a \partial_i H^0), \quad (5.19)$$

where the latter equation implies K^a to be harmonic. In this case, the $D = 3$ Cartesian rotation vector χ_i can be rewritten as

$$\chi_i = \partial_i B - 2H^0 \partial_i K^a \mathcal{Z}_a, \quad (5.20)$$

and the constraint (5.5) reads

$$H_{i0} + H_{ia} K^a + \frac{1}{2} |\epsilon_{abc}| \partial_i K^a K^b K^c H^0 - \partial_i K^a \mathcal{Z}_a - 2K^1 K^2 K^3 \partial_i H^0 = 0. \quad (5.21)$$

This solution can generate other solutions by applying the U -duality and Ehlers symmetry transformations, as considered in [1]. We leave to further future investigation the issue of existence of general multi-centered solutions of this class; for instance, in the two-centered case, such general solutions should exhibit all the duality-invariant polynomials of the “minimal degree” complete basis [27] as *independent*.

6 Conclusion

In the present investigation, we developed the most general first order formalism for multi-centered and/or under-rotating extremal BH solutions with flat three-dimensional base-space in the stu model [16, 12] of $N = 2$, $D = 4$ ungauged Maxwell-Einstein supergravity. As mentioned, this is a universal sector of all $N \geq 2$ -extended supergravity theories with symmetric (vector multiplets’) scalar manifold, and which admit an uplift to $D = 5$ dimensions.

Our procedure, which generalizes the one exploited for the simpler t^3 model in [14], sets the non-supersymmetric (composite non-BPS [1] and almost BPS [3, 1]) classes of multi-centered solutions on the same footing of the well known BPS class [8, 9], thus allowing for a unified framework for the study of the flow dynamics of scalar fields.

We developed a tree-dimensional Cartesian formalism in which the effective BH potential [18], as well as the scalar-dependent central and matter charges and the first order “fake” superpotential [10], are generalized in a not necessarily axisymmetric framework (*cfr.* (2.18), (2.24), (2.25) and (3.4)), which thus allows to handle also (under-)rotating and multi-centered BH solutions.

Then, by extending some spatially asymptotical expressions all along the flow, we derived systems of partial differential equations for the α -phases describing each class of multi-centered systems. The corresponding counting, supplemented by an algebraic constraint (and by a further condition in the almost-BPS class) is consistent with the existence of two “flat” directions along the non-rotating sigle-centered stu non-BPS flow [12, 23].

The consistency of the systems of partial differential equations on which the first order formalism is based has then been checked also by retrieving known solutions, in the single-centered limit (such as the Rasheed-Larsen non-BPS Kaluza-Klein BH [25, 26] in the composite non-BPS class) as well as in the multi-centered case (various solutions from [9, 1, 2] have been obtained as particular solutions).

Consistent with its nilpotent orbit characterization [1] as well as with the results obtained in [14] for the t^3 model, the almost BPS class [3, 1] exhibits the most involved structure: indeed, it is described by four independent phases α_0 and α_a ($a = 1, 2, 3$), and in this class the consistency of Maxwell equations further imposes the constraint (5.7), whose two-fold nature gives rise to two sub-branches. Such a split can be traced back to the possibility to have both BPS and non-BPS BH centers in this class, which thus stands on a different footing with respect to the BPS and composite non-BPS classes, in which the BH centers are of the same type.

As mentioned in Secs. 4 and 5, we leave for future investigation the issue of existence of completely general solutions (with all electric-magnetic charges switched on in each BH center) in the composite non-BPS *and/or* almost BPS classes. In the two-centered case, such general solutions would exhibit all independent duality-invariant polynomials of the complete “minimal degree” basis [27, 22, 30]. Moreover, it would be interesting to analyze the issue of the fixed distance among the BH centers in the non-supersymmetric systems [4, 1, 2], also in relation to the existence of walls of marginal stability.

It is here worth pointing out that we considered only multi-centered configurations with “large” BH centers, namely with a well-defined near-horizon geometry (and thus, scalar dynamics) already at the Einstein (two-derivatives) level; at the moment, it is not clear whether some solutions with “small” BH centers (cfr. *e.g.* [6, 31]) can be obtained in this framework (for an analysis in the two-centered case at the level of duality invariants, see [22]).

The possible extension of our approach to multi-centered systems with *non-flat* three-dimensional base-space is of utmost interest, as well; within this framework, one should recover solutions found *e.g.* in [4]. It would also be interesting to consider generalizations to *over-rotating* (single- and multi-centered) BH solutions (for recent advances on first order formalism, see *e.g.* [32]).

Our approach could also be applied to the multi-centered interacting non-BPS solutions of maximal $N = 8$, $D = 4$ supergravity recently considered *e.g.* in [2], and the issue of truncability to the universal *stu* sector is also worth being investigated in greater detail.

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A Notations and Useful Formulæ

We here recall some useful notations and formulæ, used throughout the present investigation.

In a *generic* special Kähler geometry, the symmetric real matrix M , which firstly occurred in our treatment in Eq. (2.7), is constructed from the coupling matrices $\mu_{\Lambda\Sigma}$ and $\nu_{\Lambda\Sigma}$ as follows [19, 29]

$$M_{\alpha\beta} = \begin{pmatrix} \mu_{\Lambda\Sigma} + \nu_{\Lambda\Lambda'}\mu^{\Lambda'\Sigma'}\nu_{\Sigma'\Sigma} & \nu_{\Lambda\Lambda'}\mu^{\Lambda'\Sigma} \\ \mu^{\Lambda\Lambda'}\nu_{\Lambda'\Sigma} & \mu^{\Lambda\Sigma} \end{pmatrix},$$

and it is symplectic:

$$\Omega^{\alpha\beta}M_{\beta\gamma} = -M^{\alpha\beta}\Omega_{\beta\gamma}$$

with respect to the skew-symmetric symplectic metric

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & -\delta_{\Lambda}^{\Sigma} \\ \delta_{\Sigma}^{\Lambda} & 0 \end{pmatrix}, \quad \Omega^{\alpha\beta} = \begin{pmatrix} 0 & \delta_{\Sigma}^{\Lambda} \\ -\delta_{\Lambda}^{\Sigma} & 0 \end{pmatrix},$$

which allows one to define the *symplectic product* of two vectors as:

$$\langle A, B \rangle \equiv A^{\alpha}\Omega_{\alpha\beta}B^{\beta} = A_{\Lambda}B^{\Lambda} - A^{\Lambda}B_{\Lambda}.$$

The inverse of the coupling matrix can be easily calculated to read

$$M^{\alpha\beta} = \begin{pmatrix} \mu^{\Lambda\Sigma} & -\mu^{\Lambda\Lambda'}\nu_{\Lambda'\Sigma} \\ -\nu_{\Lambda\Lambda'}\mu^{\Lambda'\Sigma} & \mu_{\Lambda\Sigma} + \nu_{\Lambda\Lambda'}\mu^{\Lambda'\Sigma'}\nu_{\Sigma'\Sigma} \end{pmatrix}.$$

The special geometry sections are normalized as

$$\langle V, \bar{V} \rangle = -i, \quad \langle V, D_a V \rangle = 0, \quad \langle D_a V, \bar{D}_{\bar{a}} \bar{V} \rangle = iG_{a\bar{a}}.$$

The coupling matrix M satisfy the following identity

$$\frac{1}{2}(M_{\alpha\beta} - i\Omega_{\alpha\beta}) = -(\Omega_{\alpha\alpha'}\bar{V}^{\alpha'}) (\Omega_{\beta\beta'}V^{\beta'}) - (\Omega_{\alpha\alpha'}D_a V^\alpha) G^{a\bar{a}} (\Omega_{\beta\beta'}\bar{D}_{\bar{a}}\bar{V}^{\beta'}).$$

The C -tensor of special Kähler geometry and the metric of the vector multiplets' scalar manifold enter the basic relations

$$D_a D_b V = iC_{abc}G^{c\bar{c}}\bar{D}_{\bar{c}}\bar{V}, \quad D_a \bar{D}_{\bar{b}} \bar{V} = G_{a\bar{b}}\bar{V}.$$

In the stu model

$$F = D_{abc}z^a z^b z^c = stu,$$

the Vielbein and its inverse (we maintain here the underlining of flat scalar indices)

$$E_a^{\underline{a}} = \text{diag}\left(\frac{1}{s - \bar{s}}, \frac{1}{t - \bar{t}}, \frac{1}{u - \bar{u}}\right), \quad E_{\underline{a}}^a = \text{diag}(s - \bar{s}, t - \bar{t}, u - \bar{u})$$

satisfy the usual definition

$$G_{a\bar{a}} = E_a^{\underline{a}} \delta_{\underline{a}\underline{\bar{a}}} \bar{E}_{\bar{a}}^{\underline{\bar{a}}}, \quad \delta_{\underline{a}\underline{\bar{a}}} = \text{diag}(1, 1, 1).$$

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